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Nongeneric SUSY in Spinning NUT–Kerr–Newman Space–Time

Mainuddin Ahmed[1] and M. Hossain Ali[1]

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The supersymmetric extension of the NUT–Kerr–Newman (NUT–KN) space–time is investigated. Along with four standard supersymmetries, this type of space–time admits fermionic symmetry generated by the square root of the bosonic constant of motion except the Hamiltonian. Such a new supersymmetry corresponds to the Killing–Yano tensor, which plays an important role in solving various field equations in this space–time.

1. INTRODUCTION

Spinning particles can be described by a pseudo-classical mechanics model in which anticommuting c-numbers characterize the spin degrees of freedom [1–6]. In ref. 7 the symmetries of space–time have been systematically investigated in terms of the motion of pseudo-classical spinning point particles described by the $d = 1$ supersymmetric extension of the usual relativistic point particle [1–5]. The general relations between space–time symmetries and the motion of spinning point particles have been studied explicitly in refs. 8–10. These methods may be applied to NUT–KN space–time, which is a stationary and axisymmetric solution of the combined Einstein–Maxwell equations. The completely integrability of particle motion in this space–time demands the existence of a nontrivial Stackel-type Killing tensor $K_{\mu\upsilon}$ [11–13], which give rise to the associated constant of motion

$$Z = \tfrac{1}{2}K^{\mu\upsilon}p_{\mu}p_{\upsilon} \tag{1}$$

quadratic in the four-momentum p_{μ}. This constant of motion forms the maximal number of constants of motion along with the other three well-known constants of motion: the energy

[1] Department of Mathematics, Rajshahi University, Rajshahi- 6205, Bangladesh.

$$E = -K^{\mu} p_{\mu} \tag{2}$$

coming from the time translation invariance generated by the Killing field K^{μ}, the angular momentum

$$J = M^{\mu} p_{\mu} \tag{3}$$

coming from the axial symmetry generated by the Killing field M^{μ}, and the Hamiltonian

$$H = \tfrac{1}{2} g^{\mu\upsilon} p_{\mu} p_{\upsilon} \tag{4}$$

Furthermore, the separability of various field equations, e.g., the Dirac equation [14], in the NUT–KN space–time has a direct consequence of the existence of the Killing–Yano tensor $f_{\mu\upsilon}$ [15], which is defined as an antisymmetric second-rank tensor satisfying the following Penrose–Floyd equation [16]:

$$D_{(\mu} f_{\upsilon)\lambda} = 0 \tag{5}$$

This Killing–Yano 2-form $f_{\mu\upsilon}$ is a square root of the Stackel–Killing tensor $K^{\mu\upsilon}$:

$$K^{\mu}_{\upsilon} = f^{\mu}_{\lambda} f^{\lambda}_{\upsilon} \tag{6}$$

Here indices are raised and lowered with the help of the space–time metric $g_{\mu\upsilon}$ and its inverse.

Recently, Gibbons *et al.* [7] have been able to show by considering supersymmetric particle mechanics that the Killing–Yano tensor can be understood as an object generating a "nongeneric" supersymmectry, i.e., a supersymmetry appearing only in a specific space–time.

In this paper, we investigate the nongeneric supersymmetry in NUT–KN space–time and discuss the constant of motion associated with it.

The organization of this paper is as follows. In Section 2 we summarize the formulation of pseudo-classical spinning point particles in an arbitrary background space–time. In Section 3 we express the general relation between symmetries, supersymmetries, and constants of motion associated with these equations. In Section 4 we describe the extra supersymmetries and their algebras. This type of supersymmetry depends on the existence of a second-rank tensor field $f_{\mu\upsilon}$ which is referred to as an *f*-symbol. In Section 5 the general properties of *f*-symbols and their relations to Killing–Yano tensors are given. In Section 6 we investigate the extra supersymmetry and the exact form of the constants of motion in the NUT–KN space–time. Finally, Section 7 is devoted to conclusions.

2. THE PSEUDO-CLASSICAL DESCRIPTION OF SPINNING PARTICLES

The pseudo-classical limit of the Dirac theory of a spin-1/2 fermion in curved space–time has recently been described by the supersymmetric extension of the ordinary relativistic point particle [1–6]. The corresponding configuration space is spanned by the real position variables $x^{\mu}(\tau)$ and the Grassmann-valued spin variables $\psi^{(a)}(\tau)$, where μ, $a = 1, \ldots, d$, with d the dimension of space–time. Greek and Latin indices refer to world and Lorentz indices, respectively, and are converted into each other by the vielbein $e^{a}_{\mu}(x)$ and its inverse $e^{\mu}_{a}(x)$. The world-line parameter τ is the invariant proper time:

$$c^2 d\tau^2 = -g_{\mu\upsilon}(x)\, dx^{\mu}\, dx^{\upsilon} \tag{7}$$

We choose units such that $c = 1$.

We start with the Lagrangian

$$L = \frac{1}{2} g_{\mu\upsilon}\dot{x}^{\mu}\dot{x}^{\nu} + \frac{1}{2} i\eta_{ab}\psi^{a} \frac{D\psi^{b}}{D\tau} \tag{8}$$

where η_{ab} is the flat-space–time (Minkowski) metric. The overdot here and in the following represents a derivative with respect to τ, and the covariant derivative of the spin variable is

$$\frac{D\psi^{a}}{D\tau} = \dot{\tau}^{a} - \dot{x}^{\upsilon}\omega^{a}_{\mu b}\psi^{b} \tag{9}$$

where $\omega^{a}_{\mu b}$ is the spin connection. Since our Lagrangian is a gauge-fixed one, we have to add appropriate constraints to fix the dynamics completely. We impose the condition expressed by Eq. (7), which is equivalent to the mass-shell condition, along with the restriction that spin be spacelike:

$$Q \equiv e_{\mu a}\dot{x}^{\mu}\psi^{a} = 0 \tag{10}$$

These supplementary constraints have to be invariant under the transformation generated through the appropriate Poisson–Dirac bracket with the equations of motion derived from the above Lagrangian [9, 10]. However, in our formulation of spinning particle dynamics these conditions are only to be imposed after solving the equations of the theory.

The solutions of the Euler–Lagrange equations derived from the Lagrangian (8) can be considered as generalizations of the concept of geodesics to spinning space spanned by (x^{μ}, ψ^{a}). The geodesics which represent the world lines of the physical spinning particles are then obtained with the help of the supplementary conditions. The classical equations of the theory can be cast in the following form [7]:

$$\frac{D^2x^\mu}{D\tau^2} = -\frac{1}{2}\, i\psi^a\psi^b R^\mu_{ab\upsilon}\dot{x}^\nu, \qquad \frac{D\psi^a}{D\tau} = 0 \tag{11}$$

Since the conjugate momenta are

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\upsilon}\dot{x}^\upsilon - \frac{1}{2}\, i\omega_{\mu ab}\psi^a\psi^b$$

$$\pi_a = \frac{\partial L}{\partial \dot{\psi}^a} = -\frac{1}{2}\, i\psi_a \tag{12}$$

the second-class constraint for π_a yields the following Poisson–Dirac bracket:

$$\{F, G\} = \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial G}{\partial x^\mu} + i(-1)^{a_F}\frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \psi_a} \tag{13}$$

where a_F is the Grassmann parity of F: $a_F = (0, 1)$ for $F = $ (even, odd). With this bracket the canonical relations

$$\{x^\mu, p_\upsilon\} = \delta^\mu_\upsilon, \qquad \{\psi^a, \psi^b\} = -i\eta^{ab} \tag{14}$$

can be checked. The theory gives the canonical Hamiltonian in the form

$$H = \tfrac{1}{2} g^{\mu\upsilon}(p_\mu + \omega_\mu)(p_\upsilon + \omega_\upsilon) \tag{15}$$

with $\omega_\mu = (1/2)i\omega_{\mu ab}\psi^a\psi^b$. The time evolution of any function $F(x, p, \psi)$ is given by

$$\frac{dF}{d\tau} = \{F, H\} \tag{16}$$

Equations (13)–(16) describe the canonical formulation of the theory. Since this formulation loses manifest covariance, we introduce the covariant momentum

$$\Pi_\mu \equiv p_\mu + \omega_\mu = g_{\mu\upsilon}\dot{x}^\upsilon \tag{17}$$

With this variable, the bracket (13) becomes

$$\{F, G\} = (\mathcal{D}_\mu F)\frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu}(\mathcal{D}_\mu G) - R_{\mu\upsilon}\frac{\partial F}{\partial \Pi_\mu}\frac{\partial G}{\partial \Pi_\upsilon}$$

$$+ i(-1)^{a_F}\frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \psi_a} \tag{18}$$

where we have used the phase-space covariant derivative operator

$$\mathcal{D}_\mu F = \partial_\mu F + \Gamma^\lambda_{\mu\upsilon}\Pi_\lambda \frac{\partial F}{\partial \Pi_\upsilon} + \omega^a_{\mu\ b}\omega^b \frac{\partial F}{\partial \psi^a} \tag{19}$$

and the spin-valued Riemann tensor

$$R_{\mu\upsilon} = \frac{1}{2} i \psi^a \psi^b R_{ab\mu\upsilon} \tag{20}$$

The Hamiltonian now takes the form

$$H = \frac{1}{2} g^{\mu\upsilon} \Pi_\mu \Pi_\upsilon \tag{21}$$

The dynamical equation (16) remains the same. The constraints (7) and (10) become

$$2H = g^{\mu\upsilon} \Pi_\mu \Pi_\upsilon = -1, \qquad Q = \Pi, \quad \psi = 0 \tag{22}$$

Since these are not compatible with the Poisson–Dirac brackets in general, they are to be imposed only after solving the equations of the theory. We see, however, that

$$\{Q, H\} = 0 \tag{23}$$

As the Hamiltonian itself is trivially conserved, Eq. (23) demands that the values of H and Q [given in (22)] are preserved in time, and the physical conditions imposed on the theory are consistent with the equations of motion (see Rietdijk [9]).

3. SYMMETRIES AND CONSTANTS OF MOTION

The theory of a spinning-particle model possesses a number of symmetries which are very useful in solving the equations of motion explicitly [10] because of their connection with constants of motion via Noether's theorem. In general, these symmetries can be divided into two classes, generic and nongeneric symmetries. The first kind exist for any space–time metric $g_{\mu\upsilon}(x)$, while the latter type depends on the explicit form of the metric. The theory described by the Lagrangian (8) admits four generic symmetries [8, 9], two of which are proper-time translations generated by the Hamiltonian H, and supersymmetry generated by the supercharge Q, Eq. (22). The other two are chiral symmetry generated by the chiral charge

$$\Gamma_* = -\frac{i^{[d/2]}}{d!} \varepsilon_{a_1 \ldots a_d} \psi^{a_1} \ldots \psi^{a_d} \tag{24}$$

and dual supersymmetry generated by the dual supercharge

$$Q^* = i\{Q, \Gamma_*\} = \frac{-i^{[d/2]}}{(d-1)!} \varepsilon_{a_1 \ldots a_d} e^{\mu a_1} \Pi_\mu \psi^{a_2} \ldots \psi^{a_d} \tag{25}$$

It can be checked that $\{H, \Gamma_*\} = 0$. Then, the Jacobi identity with (23) confirms that all the above quantities are constants of motion.

We now find all functions $J(x, \Pi, \psi)$ such that

$$\{H, J\} = 0 \tag{26}$$

These functions give all symmetries including the nongeneric ones. Using the bracket (18) simplifies (26) to

$$\Pi^{\mu}\left(\mathcal{D}_{\mu}J + R_{\mu\upsilon}\frac{\partial J}{\partial \Pi_{\mu}}\right) = 0 \tag{27}$$

Then for components of J in the expansion

$$J = \sum_{n=0}^{\infty} \frac{1}{n!} J^{(n)\mu_1 \ldots \mu_n}(x, \psi)\Pi_{\mu_1 \ldots \mu_n} \tag{28}$$

we have the following generalized Killing equations

$$D_{(\mu_{n+1}}J^{(n)}_{\mu_1 \ldots \mu_n)} + \omega^{a}_{(\mu_{n+1}b}\psi^{b}\frac{\partial J^{(n)}_{\mu_1 \ldots \mu_n) \ldots}}{\partial \psi^{a}} = R_{\upsilon(\mu_{n+1}}J^{(n+1))\upsilon}_{\mu_1 \ldots \mu_n)} \tag{29}$$

where the parentheses denote full symmetrization over the indices enclosed.

Further we notice that any constant of motion J satisfies

$$\{Q, J\} = -\psi^{\mu}\left(\mathcal{D}_{\mu}J + R_{\mu\upsilon}\frac{\partial J}{\partial \Pi_{\upsilon}}\right) - ie^{\mu a}\Pi_{\mu}\frac{\partial J}{\partial \psi^{a}} \tag{30}$$

If the curvature term undergoes three contractions with the anticommuting spin variables, then with the Bianchi identity $R_{[\mu\upsilon\lambda]\kappa} = 0$, Eq. (30) can be written as

$$\{Q, J\} = -\left(\psi \cdot \mathcal{D}J + i\Pi \cdot \frac{\partial J}{\partial \psi}\right) \tag{31}$$

In particular, for $J = Q$, we obtain the usual supersymmetry algebra:

$$\{Q, Q\} = -2iH \tag{32}$$

Then, the Jacobi identity for two Q's and any constant of motion J confirms that

$$\Theta = \{Q, J\} \tag{33}$$

is a superinvariant and hence a constant of motion as well:

$$\{Q, \Theta\} = 0, \qquad \{H, \Theta\} = 0 \tag{34}$$

This result implies that constants of motion generally come in supermultiplets (J, Θ), of which the first example is the multiplet (Q, H) itself, provided that $\Theta \neq 0$.

We observe from Eq. (31) that a superinvariant is a solution of the equation

$$\psi \cdot \mathscr{D}J + i\Pi \cdot \frac{\partial J}{\partial \psi} = 0 \tag{35}$$

Expanding $J^{(n)\mu_1 \ldots \mu_n}$ of (28) in powers of ψ^a and letting the coefficients be $f_{a_1 \ldots a_m}^{(m,n)\mu_1 \ldots \mu_n}(x)$, i.e.,

$$J = \sum_{m,n=0}^{\infty} \frac{i^{[m/2]}}{m!n!} \psi^{a_1} \ldots \psi^{a_m} f_{a_1 \ldots a_m}^{(m,n)\mu_1 \ldots \mu_n}(x) \Pi_{\mu_1} \ldots \Pi_{\mu_n} \tag{36}$$

where $f^{(m,n)}$ is completely symmetric in the n upper indices $\{\mu_k\}$ and completely antisymmetric in the m lower indices $\{a_i\}$. Equation (35) gives the component equation

$$n f_{a_0 a_1 \ldots a_m}^{(m+1,n-1)(\mu_1 \ldots \mu_{n-1}} e^{\mu_n) a_0} = m D_{[a_1} f_{a_2 \ldots a_m]}^{(m-1,n)\mu_1 \ldots \mu_n} \tag{37}$$

where $D_a = e_a^{\ \mu} D_\mu$ and square brackets denote full antisymmetrization over the indices enclosed. This equation is called the generalized Penrose–Floyd equation. It is also sometimes referred to as the square root of the generalized Killing equation [7].

4. NONGENERIC SUPERSYMMETRIES

The nongeneric supersymmetry of the theory is generated by the phase-space function Q_f,

$$Q_f = J^{(1)\mu} \Pi_\mu + J^{(0)} \tag{38}$$

where $J^{(1)}(x, \psi)$ and $J^{(0)}(x, \psi)$ are independent of Π. This charge generates the supersymmetry transformation

$$\delta x^\mu = -i\varepsilon f_a^\mu \psi^a \equiv -iJ^{(1)\mu} \tag{39}$$

where the infinte simal parameter ε of the transformation is Grassmann-odd. The ansatz (38), when inserted into the generalized Killing equations (29), gives

$$J^0 = \frac{i}{3!} C_{abc}(x) \psi^a \psi^b \psi^c \tag{40}$$

where the tensors f_a^μ and C_{abc} satisfy the conditions

$$D_\mu f_{\upsilon a} + D_\upsilon f_{\mu a} = 0 \tag{41}$$

and

$$D_\mu C_{abc} = -(R_{\mu\upsilon ab} f^\upsilon_c + R_{\mu\upsilon bc} f^\upsilon_a + R_{\mu\upsilon ca} f^\upsilon_b) \tag{42}$$

Let there be N such symmetries specified by N sets of tensors (f^μ_{ia}, C_{iabc}), $i = 1, \ldots, N$. The corresponding generators will be

$$Q_i = f^\mu_{ia} \Pi_\mu \psi^a + \frac{i}{3!} C_{iabc} \psi^a \psi^b \psi^c \tag{43}$$

Obviously, for $f^\mu_a = e^\mu_a$ and $C_{abc} = 0$, the supercharge (22) is precisely of this form. It is therefore convenient to assign the index $i = 0$: $Q = Q_0$, $e^\mu_a = f^\mu_{0a}$, etc, when we refer to the quantities defining the standard supersymmetry.

The covariant form (18) of Poisson–Dirac brackets gives the following algebra for the conserved charges Q_i:

$$\{Q_i, Q_j\} = -2Z_{ij} \tag{44}$$

where

$$Z_{ij} = \tfrac{1}{2} K^{\mu\upsilon}_{ij} \Pi_\mu \Pi_\upsilon + I^\mu_{ij} \Pi_\mu + G_{ij} \tag{45}$$

and

$$K^{\mu\upsilon}_{ij} = \tfrac{1}{2}(f^\mu_{ia} f^{\upsilon a}_j + f^\upsilon_{ia} f^{\mu a}_j) \tag{46}$$

$$\begin{aligned} I^\mu_{ij} &= \tfrac{1}{2} i\psi^a\psi^b I^\mu_{ijab} \\ &= \tfrac{1}{2} i\psi^a\psi^b \left(f^\upsilon_{ib} D_\upsilon f^\mu_{ja} + f^\upsilon_{jb} D_\upsilon f^\mu_{ia} + \tfrac{1}{2} f^{\mu c}_i C_{jabc} + \tfrac{1}{2} f^{\mu c}_j C_{iabc} \right) \end{aligned} \tag{47}$$

$$\begin{aligned} G_{ij} &= -\tfrac{1}{4} \psi^a\psi^b\psi^c\psi^d G_{ijabcd} \\ &= -\tfrac{1}{4} \psi^a\psi^b\psi^c\psi^d \left(R_{\mu\upsilon ab} f^\mu_{ic} f^\upsilon_{jd} + \tfrac{1}{2} C^{\ e}_{iab} C_{jcde} \right) \end{aligned} \tag{48}$$

We note that $K_{ij\mu\upsilon}$ is a symmetric Killing tensor of second rank:

$$D_{(\lambda} K_{ij\mu\upsilon)} = 0 \tag{49}$$

I^μ_{ij} is the corresponding Killing vector:

$$\begin{aligned} \mathcal{D}_{(\mu} I_{ij\upsilon)} &= \tfrac{1}{2} i\psi^a\psi^b D_{(\mu} I_{ij\upsilon)ab} \\ &= \tfrac{1}{2} i\psi^a\psi^b R_{ab\lambda(\mu} K_{ij\upsilon)}{}^\lambda \end{aligned} \tag{50}$$

and G_{ij} is the corresponding Killing scalar:

$$\mathscr{D}_\mu G_{ij} = -\tfrac{1}{4}\psi^a\psi^b\psi^c\psi^d D_\mu G_{ijabcd}$$
$$= \tfrac{1}{2} i\psi^a\psi^b R_{ab\lambda\mu} I^\lambda_{ij} \tag{51}$$

The functions Z_{ij} satisfy the generalized Killing equations and their bracket with the Hamiltonian vanishes. Hence, they are constants of motion. For $i = j = 0$, Eq. (43) gives the usual supersymmetry algebra (32). If i or j is not equal to zero, Z_{ij} corresponds to new bosonic symmetries unless $K^{\mu\upsilon}_{ij} = \lambda_{(ij)} g^{\mu\upsilon}$ with $\lambda_{(ij)}$ a constant (may be zero). Then the corresponding Killing vector I^μ_{ij} and Killing scallar G_{ij} disappear identically. Further, if $\lambda_{(ij)} \neq 0$, the corresponding supercharges are almost the Hamiltonian and hence there exists a second supersymmetry of the standard type. Thus the theory admits an N-extended supersymmetry with $N \geq 2$. Again, if we have a second independent Killing tensor $K^{\mu\upsilon}$ not proportional to $g^{\mu\upsilon}$, there exists a genuine new type of supersymmetry.

Following (34), we see that $\{Q_i, Q\} = 0$ and hence Q_i is a superinvariant. The condition for this is

$$K^{\mu\upsilon}_{0i} = f^\mu{}_a e^{\upsilon a} + f^\upsilon{}_a e^{\mu a} = 0 \tag{52}$$

As the Z_{ij} are symmetric in (ij), we can diagonalize them. This provides the algebra

$$\{Q_i, Q_j\} = -2i\delta_{ij} Z_i \tag{53}$$

where Z_i are $N + 1$ conserved bosonic charges. If all Q_i satisfy condition (52), the first of these diagonal charges (with $i = 0$) is the Hamiltonian: $Z_0 = H$.

5. PROPERTIES OF THE *f*-SYMBOLS

The f-symbol is the second-rank tensor

$$f_{\mu\upsilon} = f_{\mu a} e^a_\upsilon \tag{54}$$

Condition (41) then gives

$$D_\upsilon f_{\lambda\mu} + D_\lambda f_{\upsilon\mu} = 0 \tag{55}$$

This implies that the divergence on the first index of the f-symbol vanishes:

$$D_\upsilon f^\upsilon_\mu = 0 \tag{56}$$

On contraction, Eq. (55) gives

$$D_\upsilon f^\upsilon_\mu = -\partial_\mu f^\upsilon_\upsilon \tag{57}$$

and hence the f-symbol will also be divergenceless on the second index if and only if its trace is constant:

$$D_\upsilon f^\upsilon_\mu = 0 \Leftrightarrow f^\mu_\mu = \text{constant} \tag{58}$$

Since $g_{\mu\upsilon}$ is a trivial solution of Eq. (55), we may subtract it from the f-symbol. Then, taking the constant equal to zero, f can be made traceless.

From Eq. (46) with $f^\mu_{0a} = e^\mu_a$, the symmetric part of the ith f-symbol is the tensor

$$S_{\mu\upsilon} \equiv K_{i0\mu\upsilon} = \tfrac{1}{2}(f_{\mu\upsilon} + f_{\upsilon\mu}) \tag{59}$$

which satisfies the generalized Killing equation

$$D_{(\mu}S_{\upsilon\lambda)} = 0 \tag{60}$$

Also, the antisymmetric part can be constructed as

$$B_{\mu\upsilon} = -B_{\upsilon\mu} = \tfrac{1}{2}(f_{\mu\upsilon} - f_{\upsilon\mu}) \tag{61}$$

satisfying the condition

$$D_\upsilon B_{\lambda\mu} + D_\lambda B_{\upsilon\mu} = D_\mu S_{\upsilon\lambda} \tag{62}$$

Equations (55) and (62) indicate that f is completely symmetric if it is covariantly constant.

We now consider the case in which the f-symbol is completely antisymmetric: $f_{\mu\upsilon} = B_{\mu\upsilon}$. Then condition (52) implies that the supercharge Q_f will anticommute with ordinary supersymmetry Q in the sense of Poisson–Dirac brackets. Also, Eq. (58) is trivially satisfied in this case. It is possible to say much more about the explicit form of the quantities introduced above.

Let us consider that the symmetric part of a certain tensor $f_{i\mu\upsilon}$ vanishes:

$$S^{\mu\upsilon}_i = K^{\mu\upsilon}_{i0} = 0 \tag{63}$$

Then the corresponding Killing vector and tensor also become zero. Thus, for this particular i, $Z_{i0} = 0$ which implies that Q_i is superinvariant, i.e.,

$$\{Q_i, Q\} = 0 \tag{64}$$

For antisymmetric $f_{\mu\upsilon}$, Eq. (55) gives

$$D_\upsilon B_{\lambda\mu} = -D_\lambda B_{\upsilon\mu} \tag{65}$$

Since $B_{\mu\upsilon}$ is antisymmetric, Eq. (65) implies that the gradient is completely antisymmetric:

$$D_\mu B_{\upsilon\lambda} = D_{[\mu}B_{\upsilon\lambda]} \equiv H_{\mu\upsilon\lambda} \tag{66}$$

The second covariant derivative of $f_{\mu\upsilon}$, with commutation of the derivatives and application of Eq. (55) gives the identity

$$D_\mu D_\upsilon f_{\lambda\kappa} = R^{\sigma}_{\upsilon\lambda\mu} f_{\sigma\kappa} + \tfrac{1}{2}(R^{\sigma}_{\upsilon\lambda\kappa} f_{\mu\sigma} + R^{\sigma}_{\mu\lambda\kappa} f_{\upsilon\sigma} - R^{\sigma}_{\mu\upsilon\kappa} f_{\lambda\sigma}) \tag{67}$$

For antisymmetric $f_{\mu\upsilon}$, Eq. (67) implies

$$D_\mu H_{\upsilon\lambda\kappa} = \tfrac{1}{2}(R^{\sigma}_{\upsilon\lambda\mu} f_{\sigma\kappa} + R^{\sigma}_{\lambda\kappa\mu} f_{\sigma\upsilon} + R^{\sigma}_{\kappa\upsilon\mu} f_{\sigma\lambda}) \tag{68}$$

Comparing Eq. (68) with Eq. (42), we find that

$$-\tfrac{1}{2}C_{abc} = H_{abc} = e^{\mu}_{a} e^{\upsilon}_{b} e^{\lambda}_{c} H_{\mu\upsilon\lambda} \tag{69}$$

modulo a covariantly constant term. Equation (37) with $m = 2$, $n = 1$ gives this result. The covariantly constant three-index tensor C_{abc} provides another independent symmetry corresponding to the Killing vector

$$I_\mu = \tfrac{1}{2} i\psi^a \psi^b e^{c}_{\mu} C_{abc} \tag{70}$$

In order to construct a constant of motion, a particular solution of Eq. (42) is needed. Thus the covariantly constant term can be chosen to vanish. If $D_\mu C_{abc} = 0$, then

$$\mathcal{D}_\mu I_\upsilon = 0 \tag{71}$$

and automatically I_μ satisfies the generalized Killing equation.

According to Eq. (63), $K^{\mu\upsilon}_{0i} = 0$, and $C_{0abc} = 0$ identically. Hence the right-hand side of Eq. (47) becomes

$$I_{i0\mu\upsilon\lambda} \equiv I_{i0\mu ab} e^{a}_{\upsilon} e^{b}_{\lambda} = D_\lambda B_{i\mu\upsilon} + \tfrac{1}{2} C_{i\mu\upsilon\lambda} = 0 \tag{72}$$

where the last equality follows from Eq. (69). Also, the Killing scalar $G_{i0} = 0$, because of the cyclic Bianchi identity for $R_{\mu\upsilon\lambda\kappa}$ and $C_{0abc} = 0$. Thus Eq. (64) is justified.

6. SPINNING NUT–KERR–NEWMAN SPACE–TIME

We now apply the results of the previous sections to the motion of spinning particles in NUT–Kerr–Newman space–time and investigate the existence of a new type of supersymmetry. The NUT–KN space–time has the metric

$$\begin{aligned} ds^2 &= \rho^2(\Delta^{-1} dr^2 + d\theta^2) + \rho^{-2} \sin^2\theta [a\, dt - (r^2 + a^2)\, d\phi]^2 \\ &\quad - \Delta\rho^{-2}\left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a}\right) d\phi\right]^2 \end{aligned} \tag{73}$$

with

$$\Delta = r^2 + a^2 - n^2 - 2Mr + q^2$$

$$\rho^2 = r^2 + (n - a\cos\theta)^2$$

where M is the mass, q the charge, $a\ (=J/M)$ the specific angular momentum, and n the NUT parameter. The electromagnetic field in this space–time is given by

$$F = q\rho^{-4}[r^2 - (n - a\cos\theta)^2]dr \wedge \left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a}\right)d\phi\right] + 2q\rho^{-4}(n - a\cos\theta)r\sin\theta\, d\theta \wedge [a\,dt - (r^2 + a^2)\,d\phi] \tag{74}$$

The NUT–KN space–time admits two independent Stackel–Killing tensors, as was found in ref. 12. These are the metric tensor $g_{\mu\nu}$ and the Stackel–Killing tensor $K_{\mu\nu}$, and their corresponding conserved quantities are the Hamiltonian H and the quantity Z. For supersymmetric extension of this result, we use the antisymmetric Killing–Yano tensor $f_{\mu\nu}$ found by Penrose and Floyed [16], which satisfies Eq. (55) and is the f-symbol of the double vector f^a_μ as defined in (54). The Stackel–Killing tensor $K_{\mu\nu}$ is exactly the covariant square of this tensor. Then the new supersymmetry in spinning NUT–KN space–time is obtained from a supercharge as given in Eq. (43) with the f-symbol and a corresponding three-index tensor C_{abc} given by Eq. (69).

We first derive the explicit expression for the new supercharge. Using this, we then obtain the killing vector I_μ and the Killing scalar G which correspond to the Stackel–Killing tensor $K_{\mu\nu}$ and define the conserved charge Z.

The Killing–Yano tensor in spinning NUT–KN space–time is obtained from [16]

$$\frac{1}{2} f_{\mu\nu}\, dx^\mu \wedge dx^\upsilon = -(n - a\cos\theta)\, dr \wedge \left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a}\right)d\phi\right] - r\sin\theta\, d\theta \wedge [a\,dt - (r^2 + a^2)\,d\phi] \tag{75}$$

The vielbein $e^a_\mu(x)$ corresponding to the metric (73) has the following expressions:

$$\begin{aligned}
e^0_\mu\, dx^\mu &= -\frac{\sqrt{\Delta}}{\rho}\left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a}\right)d\phi\right]\\
e^1_\mu\, dx^\mu &= \frac{\rho}{\sqrt{\Delta}}\, dr\\
e^2_\mu\, dx^\mu &= \rho d\theta\\
e^3_\mu\, dx^\mu &= -\frac{\sin\theta}{\rho}[a\,dt - (r^2 + a^2)\,d\phi]
\end{aligned} \tag{76}$$

We thus get the following components of $f^a_\mu(x)$:

$$
\begin{aligned}
f^0_\mu dx^\mu &= -\frac{\rho}{\sqrt{\Delta}}(n - a\cos\theta)\,dr \\
f^1_\mu dx^\mu &= \frac{\sqrt{\Delta}}{\rho}(n - a\cos\theta)\left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a}\right)d\phi\right] \\
f^2_\mu dx^\mu &= \frac{r\sin\theta}{\rho}[a\,dt - (r^2 + a^2)\,d\phi] \\
f^3_\mu dx^\mu &= \rho r\,d\theta
\end{aligned} \tag{77}
$$

and indeed, this $f^a_\mu(x)$ satisfies Eq. (41). To get a conserved quantity we now need to find $C_{abc}(x)$. Using Eq. (69), its components are given as follows:

$$
C_{012} = \frac{2a\sin\theta}{\rho}, \qquad C_{013} = 0, \qquad C_{023} = 0, \qquad C_{123} = -\frac{2\sqrt{\Delta}}{\rho} \tag{78}
$$

With the help of the quantities derived in Eqs. (77) and (78) we obtain from Eq. (43) the generator Q_f of the new supersymmetry for spinning NUT–KN space–time. From Eqs. (46)–(48) we respectively construct the Killing tensor, vector, and scalar. The results are

$$
\begin{aligned}
&K_{\mu\upsilon}(x)\,dx^\mu\,dx^\upsilon \\
&\quad= \frac{-\rho^2(n - a\cos\theta)^2}{\Delta}dr^2 + \frac{\Delta(n - a\cos\theta)^2}{\rho^2} \\
&\qquad\times\left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a}\right)d\phi\right]^2 \\
&\qquad+ \frac{r^2\sin^2\theta}{\rho^2}[a\,dt - (r^2 + a^2)\,d\phi]^2 + \rho^2 r^2 d\theta^2
\end{aligned} \tag{79}
$$

$$
\begin{aligned}
I_\mu(x)\,dx^\mu = {}& \frac{-2i}{\rho^2}(r\sin\theta\,\psi^1 + \sqrt{\Delta}\cos\theta\,\psi^2)(a\sin\theta\,\psi^0 - \sqrt{\Delta}\,\psi^3) \\
&\times a\,dt - (r^2 + a^2)\,d\phi] \\
&- i\sqrt{\Delta}\cos\theta\,\psi^2(a\sin\theta\,\psi^0 - \sqrt{\Delta}\,\psi^3)\,d\phi \\
&+ i\sqrt{\Delta}\,(r\sin\theta\,\psi^1 + \sqrt{\Delta}\cos\theta\,\psi^2)\psi^3\,d\phi \\
&+ \frac{ia\sin\theta}{\sqrt{\Delta}}[r\psi^0\psi^3 - (n - a\cos\theta)\,\psi^1\psi^2]\,dr \\
&- i\sqrt{\Delta}\,[(n - a\cos\theta)\psi^0\psi^3 + r\psi^1\psi^2]\,d\theta
\end{aligned} \tag{80}
$$

$$G = \frac{2q(n - a\cos\theta)}{\rho^2}\psi^0\psi^1\psi^2\psi^3 \tag{81}$$

The expression for Q_f and (79)–(81) then define the conserved charge $Z = 1/2\ i\{Q_f, Q_f\}$.

7. CONCLUSIONS

The spinning space–time is the extension of the ordinary space–time with the antisymmetric Grassmann variables to describe the spin degrees of freedom. The standard antisymmetric spin tensor S^{ab}, which appears in the definition of the generators of the local Lorentz transformations, is related to these spin variables by $S^{ab} = -i\psi^a\psi^b$. This relation makes the physical interpretation of the equations (79)–(81) more clear. Using the Dirac–Poisson brackets (13), it can be shown that these equations satisfy the $SO(3,1)$ algebra. The generators of the full Lorentz transformations are given by $M^{ab} = L^{ab} + S^{ab}$ with L^{ab} the orbital part. Likewise, the generators of other symmetries such as Z also receive a spin-dependent part. For scalar (spinless) point particles in NUT–KN space–time the Stackel–Killing tensor $K_{\mu\upsilon}$ given in (79) gives a constant of motion which for spinning point particles receives nontrivial contributions from spin. This spin-dependent part contains the Killing vector and Killing scalar computed in Section 6. The antisymmetric Killing–Yano tensor $f_{\mu\upsilon}$, which is the square root of the Stackel–Killing tensor, describes this spin-dependent part.

In refs. 15 and 17 antisymmetric f-symbols and their corresponding Killing tensors were studied in the context of obtaining solutions of the Dirac equation in nontrivial curved space–time. The Killing–Yano and Stackel–Killing tensors stated above have precise correspondence with them. The analysis given in Section 5 shows that they belong to a larger class of possible structures which generate generalized supersymmetry algebras.

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